Elliptic parameters and defining equations for elliptic fibrations on a Kummer surface

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ABSTRACT. We pose the problem to determine explicit defining equations of various elliptic fibrations on a given $K3$ surface, and study the case of the Kummer surfaces of the product of two elliptic curves.

1. Introduction

1.1. Problem setting. Let X be a $K3$ surface defined over a base field k, and let $k(X)$ denote its function field. Suppose $f : X \to \mathbf{P}^1$ is an elliptic fibration on X with a section O. Then it defines a non-constant function $u = f(x)$ $(x \in X)$, and hence an element $u \in k(X)$. We call u the *elliptic parameter* for the elliptic fibration f. (Actually u is unique only up to the linear fractional transformations, but to fix the idea, we always choose one u. Note that the subfield $k(u)$ of $k(X)$ is uniquely defined by f).

Now let E denote the generic fiber of f. Then E is an elliptic curve defined over $k(u)$ such that the function field $k(u)(E)$ is isomorphic to $k(X)$ as the extensions of k.

PROBLEM 1. *Given a $K3$ surface X/k and an elliptic fibration f, determine (i) the elliptic parameter u for f, (ii) the defining equation of the elliptic curve $E/k(u)$, and (iii) the Mordell-Weil lattice (MWL) $E(k(u))$.*

PROBLEM 2. *Given a $K3$ surface X/k, determine all the (essentially distinct) elliptic parameters.*

Problem 2 is a combination of Problem 1 and the following standard problem:

PROBLEM 3. *Given a $K3$ surface X/k, classify the elliptic fibrations $f : X \to \mathbf{P}^1$ up to isomorphisms.*

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1.2. Main results. In this paper, we focus on the case of Kummer surfaces $X = \mathrm{Km}(A)$, where $A = C_1 \times C_2$ is a product of two elliptic curves, and assume k is an algebraically closed field of characteristic different from 2.

In this case, Problem 3 has been solved by Oguiso [**8**] under the assumption

(#) C_1, C_2 are not isogenous to each other and $k = \mathbf{C}$ (the field of complex numbers).

Namely he classifies the configuration of singular fibers on such a Kummer surface X into eleven types $\mathscr{J}_1, \ldots, \mathscr{J}_{11}$, and determines the number of the isomorphism classes for each type.

Our main results can be stated as follows: we solve Problem 1 for each type of Oguiso's list (without assuming (#)), and thus solve Problem 2 under the assumption (#). More details will be given in §1.5 and 1.7 after we fix the notation and review some known cases.

1.3. Notation. By a (-2)-curve we mean a smooth rational curve on X whose self-intersection number is -2. (It is called a "nodal curve" in Oguiso [**8**].) It is known (cf. [**4**]) that all irreducible components of a reducible fiber in an elliptic fibration are (-2)-curves.

We have a configuration of twenty-four (-2)-curves on X, called the *double Kummer pencil* (see Fig. 1, cf. [**10**]). It consists of the 16 exceptional curves A_{ij} arising from the minimal resolution $X \to A/\iota_A$, plus the 8 curves F_i, G_j obtained as the image of $v_i \times C_2$ or $C_1 \times v'_j$ under the rational map $A \to S$. Here $\{v_i\}$ (or $\{v'_i\}$) denote the 2-torsion points of C_1 (resp. C_2) $(i, j \in I = \{0,1,2,3\})$, and ι_A denotes the inversion automorphism of A. These curves will be referred to as the *basic curves* below.

Suppose that the elliptic curve C_i is defined by the Legendre form

$$C_i : y_i^2 = x_i(x_i - 1)(x_i - \lambda_i) \qquad \lambda_i \neq 0, 1.$$

We order the 2-torsion points by $v_1 = (0,0), v_2 = (1,0), v_3 = (\lambda_1, 0)$, with v_0 denoting the origin of C_1; similarly for v'_j and C_2.

The function field $k(X)$ is equal to the subfield of the function field $k(A) = k(x_1, y_1, x_2, y_2)$ consisting of the elements invariant under the inversion $(x_1, y_1, x_2, y_2) \mapsto (x_1, -y_1, x_2, -y_2)$, namely we have

$$k(X) = k(x_1, x_2, t), \quad t = \frac{y_2}{y_1},$$

where x_1, x_2 and t are naturally regarded as functions on X, satisfying the relation

$$x_1(x_1 - 1)(x_1 - \lambda_1)t^2 = x_2(x_2 - 1)(x_2 - \lambda_2). \tag{1.1}$$

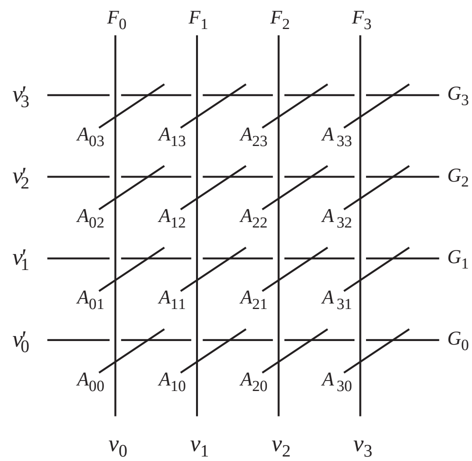

FIGURE 1. double Kummer pencil

1.4. Examples. We start from the most classical and elementary example:

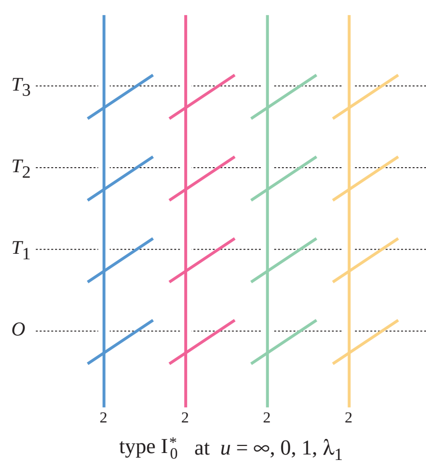

FIGURE 2. Kummer pencil (type $\mathscr{J}_4$)

EXAMPLE 1.1 (Kummer pencils). The projection of A to the first factor induces an elliptic fibration $\pi_1 : X \to \mathbf{P}^1$ with four singular fibers of type I_0^*:

$$\Phi_i = 2F_i + \sum_j A_{ij}$$

(see Fig. 2). This π_1 and the similar π_2 (obtained from the second projection) are respectively called the first or second Kummer pencil on X. The elliptic parameter for π_1 (or π_2) is obviously given by the function x_1 (resp. x_2) in $k(X)$. (This belongs to type $\mathscr{J}_4$ in [**8**], and π_1 and π_2 are the two representatives of isomorphism classes, if C_1, C_2 are not isogenous.)

The defining equation of the generic fiber over $k(x_1)$ is easily obtained (see §2.3), which is isomorphic to the constant curve C_2 over the quadratic extension $k(x_1, y_1) = k(C_1)$ of $k(x_1)$. The Mordell-Weil lattice is isomorphic to the *lattice* $\mathrm{Hom}(C_1, C_2)$ with norm $\varphi \mapsto \deg(\varphi)$ up to torsion (see [**14**, Prop.3.1]).

The next is the motivating example for studying the elliptic parameters and the problems posed in §1.1 in general.

Example 1.2 (Inose's pencils). Using the twenty-four basic curves, we can find two disjoint divisors of Kodaira type IV*. Namely, take the following divisors shown in Fig. 3:

$$\begin{cases} \Psi_1 = G_1 + G_2 + G_3 + 2(A_{01} + A_{02} + A_{03}) + 3F_0, \\ \Psi_2 = F_1 + F_2 + F_3 + 2(A_{10} + A_{20} + A_{30}) + 3G_0, \end{cases}$$

There is an elliptic fibration, called *Inose's pencil*, having these divisors as

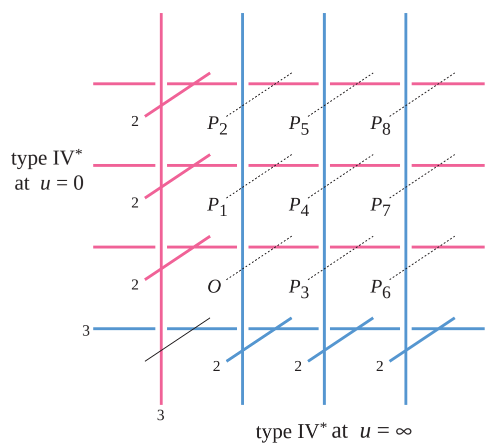

Figure 3. Inose's pencil (type $\mathscr{J}_3$)

the singular fibers over $u = 0$ and $u = \infty$, as first shown by Inose [**3**]. The elliptic parameter for this is given by the function $u = t(= y_2/y_1) \in k(X)$, and the generic fiber $E/k(t)$ is isomorphic to the cubic curve defined by the equation (1.1) in the projective plane with inhomogeneous coordinates x_1, x_2. (This belongs to type $\mathscr{J}_3$ in [**8**].)

It should be remarked that Kuwata [**6**] has succeeded in constructing, by the use of Inose's pencil, some elliptic $K3$ surfaces with high Mordell-Weil rank which have an explicit defining equation. For example, the base change $t = s^3$ gives rise to the elliptic curve $E/k(s)$ which has the highest possible rank $r = 18$ (for $k = \mathbf{C}$) provided that C_1 and C_2 are mutually isogenous but non-isomorphic elliptic curves with complex multiplications. We refer to Kuwata [**6**] and Shioda [**12**], [**14**] for more details including the defining equation of E in the Weiertrass form as well as the structure of MWL; see also §2.2.

Example 1.3. Besides the Kummer pencils (Example 1.1), the elliptic pencil on the Kummer surface $X = \mathrm{Km}(C_1 \times C_2)$ which has been studied first is perhaps the one introduced in Shioda-Inose [**10**]. It has II^*, I_0^*, I_0^* as reducible singular fibers (for general values of λ_1 and λ_2). This has type $\mathscr{J}_9$ in [**8**] (see Fig. 16). Via the base change of degree 2, it gives rise to an elliptic $K3$ surface with two II^* fibers, which plays an important role in the theory of singular $K3$ surfaces [**10**] and which has been reconsidered by Morrison [**7**] in a more general situation. It turns out that the elliptic parameter and the defining equation for this type $\mathscr{J}_9$ is the hardest case treated in this paper (see §5.3).

1.5. Results. In the following Table 1, we give a summary of the elliptic parameters and the structure of the MWL for each type $\mathscr{J}_n$, to be constructed in the subsequent sections.

The first column shows the type $\mathscr{J}_n$ of elliptic fibration following Oguiso's notation (cf. [**8**]). The second column shows the configuration of singular fibers in the *generic case*, which means that λ_1 and λ_2 are algebraically independent elements of k over $\mathbf{Q}_0$, where $\mathbf{Q}_0$ is the prime field in k. The third column shows the structure of MWL of the generic fiber E over $k(u)$, again in the generic case. The last column gives the elliptic parameter which can be used for any $\lambda_1, \lambda_2 (\neq 0, 1)$.

The explicit form of defining equations should be found in the text, since it is not suitable to tabulate here. We note that each of these defining equations has coefficients in $\mathbf{Q}_0(\lambda_1, \lambda_2)(u)$, where u is the elliptic parameter.

We see from the table that the elliptic parameters for $\mathscr{J}_n$ for $n = 1, 2, 3$ are of the form $u = t\varphi(x_1, x_2)$ with $\varphi(x_1, x_2) \in k(x_1, x_2)$, while those for $\mathscr{J}_n$ for $n > 3$ are contained in $k(x_1, x_2)$.

1.6. Basic strategy of construction. Theoretically, constructing an elliptic fibration on a $K3$ surface is to find a divisor that has the same type as a singular fiber in the Kodaira's list (cf. [**4**] [**9**]). In practice, however, we need to find two divisors, one for the fiber at $u = 0$, and the other for the

Type	Singular fibers	MWL	Elliptic parameter u
$\mathscr{J}_1$	$2\mathrm{I}_8 + 8\mathrm{I}_1$	$\mathbf{Z}^2 \oplus \mathbf{Z}/2\mathbf{Z}$	$\dfrac{tx_1}{x_2}$
$\mathscr{J}_2$	$\mathrm{I}_4 + \mathrm{I}_{12} + 8\mathrm{I}_1$	$A_2^*[2] \oplus \mathbf{Z}/2\mathbf{Z}$	$\dfrac{t(x_1-\lambda_1)(x_1-x_2)}{x_2(x_2-1)}$
$\mathscr{J}_3$	$2\mathrm{IV}^* + 8\mathrm{I}_1$	$(A_2^*[2])^2$	t
$\mathscr{J}_4$	$4\mathrm{I}_0^*$	$(\mathbf{Z}/2\mathbf{Z})^2$	x_i
$\mathscr{J}_5$	$\mathrm{I}_6^* + 6\mathrm{I}_2$	$(\mathbf{Z}/2\mathbf{Z})^2$	$\dfrac{(x_1-x_2)\big(\lambda_2(x_1-\lambda_1)+(\lambda_1-1)x_2\big)}{(\lambda_2x_1-x_2)\big(x_1-\lambda_1+(\lambda_1-1)x_2\big)}$
$\mathscr{J}_6$	$2\ \mathrm{I}_2^* + 4\mathrm{I}_2$	$(\mathbf{Z}/2\mathbf{Z})^2$	$\dfrac{x_1}{x_2}$
$\mathscr{J}_7$	$\mathrm{I}_4^* + 2\mathrm{I}_0^* + 2\mathrm{I}_1$	$\mathbf{Z}/2\mathbf{Z}$	$\dfrac{(x_2-\lambda_2)(x_1-x_2)}{(x_2-1)(\lambda_2x_1-x_2)}$
$\mathscr{J}_8$	$\mathrm{III}^* + \mathrm{I}_2^* + 3\mathrm{I}_2 + \mathrm{I}_1$	$\mathbf{Z}/2\mathbf{Z}$	$-\dfrac{(x_2-\lambda_2)(x_1-x_2)}{\lambda_2(\lambda_2-1)x_1(x_1-1)}$
$\mathscr{J}_9$	$\mathrm{II}^* + 2\mathrm{I}_0^* + 2\mathrm{I}_1$	$\{0\}$	$\dfrac{(x_2-\lambda_2)(x_1-x_2)\big(\lambda_2x_1(x_1-1)+(\lambda_1-1)(x_2-1)(\lambda_2x_1-x_2)\big)}{(x_2-1)(\lambda_2x_1-x_2)\big(\lambda_2x_1(x_1-1)+(\lambda_1-1)(x_2-\lambda_2)(x_1-x_2)\big)}$
$\mathscr{J}_{10}$	$\mathrm{I}_8^* + \mathrm{I}_0^* + 4\mathrm{I}_1$	$\{0\}$	$\dfrac{(x_2-\lambda_2)(x_1-x_2)\big((\lambda_1-1)(x_2-1)(\lambda_2x_1-x_2)+\lambda_2x_1(x_1-1)\big)}{x_2(x_2-1)(x_1-1)(\lambda_2x_1-x_2)}$
$\mathscr{J}_{11}$	$2\mathrm{I}_4^* + 4\mathrm{I}_1$	$\{0\}$	$\dfrac{x_2(x_2-\lambda_2)(x_1-x_2)}{x_1(x_2-1)(\lambda_2x_1-x_2)}$

TABLE 1. Results

fiber at $u = \infty$, to write down an actual elliptic parameter. This is where the difficulty is.

Once an elliptic parameter is found, we want to find a change of variables that converts the defining equation to a Weierstrass form. In most cases we encounter an equation of the form $y^2 =$ (quartic polynomial). We then use a standard algorithm to transform it to a Weierstrass form (see for example Cassels [**1**], or Connell [**2**]).

Some elliptic fibrations have nontrivial Mordell-Weil group. To determine the structure of Mordell-Weil lattice, we can use the method in [**11**] since we understand very well the intersection between the section and the components of singular fibers. Alternatively, we can compute the height pairing using the algorithm in [**5**] once we establish the conversion to the Weierstrass form. Note that [**11**] and [**5**] use different normalization of the height pairing, and they differ by a multiple of 2. In this article we adopt the normalization used in [**11**].

1.7. Remark. Fix a type $\mathscr{J}_n$ $(n = 1, \dots, 11)$. As noted in §1.5, each of the defining equation of $E/k(u)$ constructed in this paper has the coefficients in $\mathbf{Q}_0(\lambda_1, \lambda_2)(u)$, where u is the elliptic parameter, and λ_1 (resp. λ_2) is the Legendre parameter for C_1 (resp. C_2). Given C_i, there are in general six choices of λ_i (i.e., six different level 2-structures on C_i). We have verified that, by different choices of λ_1 or λ_2, we obtain as many nonisomorphic E's belonging to the same type $\mathscr{J}_n$, as predicted by Oguiso's result ([**8**, Table B, p. 652]), and thus solved Problem 2 when C_1 and C_2 are not isogenous. The proof for this will be omitted in this paper, but we write down the results in a single special case where we take $C_1 : y_1^2 = x_1^3 - 1$ and $C_2 : y_2^2 = x_2^3 - x_2$ (see §6).

This paper is organized as follows.

Contents

2. Elliptic parameters for $\mathscr{J}_1$, $\mathscr{J}_3$, $\mathscr{J}_4$, and $\mathscr{J}_6$

In this section we construct elliptic fibrations that have two singular fibers consisting only of the twenty-four basic curves. We use combinations of the following divisors of typical functions (cf. Examples in §1.4):

$$
\begin{aligned}
(x_1) &= 2F_1 + A_{10} + A_{11} + A_{12} + A_{13} - (2F_0 + A_{00} + A_{01} + A_{02} + A_{03}),\\
(x_2) &= 2G_1 + A_{01} + A_{11} + A_{21} + A_{31} - (2G_0 + A_{00} + A_{10} + A_{20} + A_{30}),\\
(t) &= G_1 + G_2 + G_3 + 2(A_{01} + A_{02} + A_{04}) + 3F_0\\
&\qquad - (F_1 + F_2 + F_3 + 2(A_{10} + A_{20} + A_{30}) + 3G_0).
\end{aligned}
$$

2.1. $\mathscr{J}_1$. An elliptic parameter for the type $\mathscr{J}_1$ fibration is given by

$$u = \frac{tx_1}{x_2}.$$

It is easy to verify that the divisor of u is given by

$$
\begin{aligned}
(u) = F_0 + F_1 + G_2 + G_3 + A_{02} + A_{03} + A_{12} + A_{13}\\
- (G_0 + G_1 + F_2 + F_3 + A_{20} + A_{21} + A_{30} + A_{31}),
\end{aligned}
$$

which is indicated in Fig. 4. Choosing A_{00} as the 0-section of the group

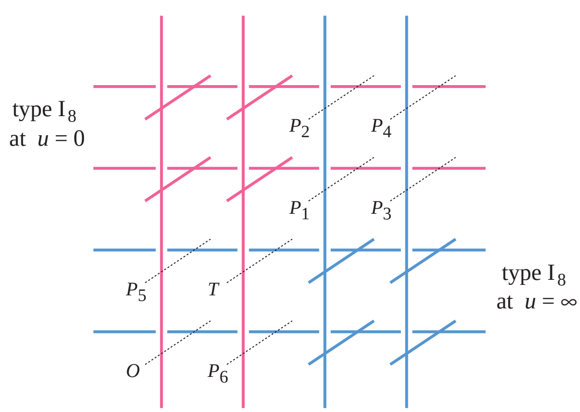

FIGURE 4. $\mathscr{J}_1$

structure, we obtain the Weierstrass equation of the elliptic fibration

$$Y^2 = X^3 + \big((\lambda_1 - 1)^2 u^4 - 2(\lambda_1 + 1)(\lambda_2 + 1)u^2 + (\lambda_2 - 1)^2\big)X^2 + 16\lambda_1\lambda_2 u^4 X,$$

where the change of variables is given by

$$X = \frac{4t^2x_1^3}{x_2}, \quad Y = \frac{4t^2x_1^3\big(t^2x_1(x_1^2 - \lambda_1) + x_2(x_2^2 - \lambda_2)\big)}{x_2^3}.$$

Its discriminant is given by

$$\Delta(u) = 2^{12}\lambda_1^2\lambda_2^2\, u^8\, d(u)d(-u),$$

where $d(u)$ is a polynomial of degree 4 in u:

$$\begin{aligned} d(u) = (\lambda_1-1)^2\,u^4 + 4(\lambda_1-1)\,u^3 \\ - 2(\lambda_1\lambda_2+\lambda_1+\lambda_2-3)\,u^2 + 4(\lambda_2-1)\,u + (\lambda_2-1)^2. \end{aligned}$$

[The discriminant of $d(u)$ vanishes if and only if $\lambda_1 = \lambda_2$. If $\lambda_1 = \lambda_2$, the elliptic fibration has two I_2 fibers for general λ_1.]

The curve A_{11} corresponds to the 2-torsion section $T = (0,0)$. The correspondence between the curves and the sections are as follows:

$$\begin{aligned} A_{22} &\leftrightarrow P_1 = \bigl(4u^2, -4u^2((\lambda_1-1)u^2+\lambda_2-1)\bigr) \\ A_{23} &\leftrightarrow P_2 = \bigl(4\lambda_2u^2, -4\lambda_2u^2((\lambda_1-1)u^2-\lambda_2+1)\bigr) \\ A_{32} &\leftrightarrow P_3 = \bigl(4\lambda_1u^2, 4\lambda_1u^2((\lambda_1-1)u^2-\lambda_2+1)\bigr) \\ A_{33} &\leftrightarrow P_4 = \bigl(4\lambda_1\lambda_2u^2, 4\lambda_1\lambda_2u^2((\lambda_1-1)u^2+\lambda_2-1)\bigr) \\ A_{01} &\leftrightarrow P_5 = \bigl(4\lambda_2, 4\lambda_2((\lambda_1+1)u^2-\lambda_2-1)\bigr) \\ A_{10} &\leftrightarrow P_6 = \bigl(4\lambda_1u^4, -4\lambda_1u^4((\lambda_1+1)u^2-\lambda_2-1)\bigr) \end{aligned}$$

These sections satisfy the following relations.

$$\begin{aligned} P_3 &= P_2 + T, & P_4 &= P_1 + T, \\ P_5 &= P_1 + P_2, & P_6 &= P_5 + T. \end{aligned}$$

The Mordell-Weil group is generated by T, P_1 and P_2 in the general case where C_1 and C_2 are not isogenous. The height matrix with respect to $\{P_1, P_2\}$ is shown to be

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2.2. $\mathscr{J}_3$**.** As we have seen in Example 1.2 (§1.4),

$$u = t$$

gives an elliptic parameter of type $\mathscr{J}_3$. We regard (1.1) as a cubic curve in x_1 and x_2 with coefficients in $k(u) = k(t)$. We choose $(x_1, x_2) = (0,0)$ as the origin of the group structure. The Weierstrass form is given by

$$\begin{aligned} Y^2 = X^3 &+ 4(\lambda_1+1)(\lambda_2+1)u^2X^2 \\ &+ 16u^4\bigl((\lambda_1\lambda_2+1)(\lambda_1+\lambda_2+1)-1\bigr)X \\ &\quad + 16u^4\bigl((\lambda_1(\lambda_1-1)u^2+\lambda_2(\lambda_2-1))^2 + 4\lambda_1\lambda_2(\lambda_1+\lambda_2)u^2\bigr). \end{aligned}$$

(This is relatively simple, but the intermediate calculations are rather complicated.) The change of variables between two forms of equations is given by

$$\begin{aligned} X &= \frac{4\big(\lambda_2(x_1-1)(x_1-\lambda_1)+\lambda_1(x_2-1)(x_2-\lambda_2)-\lambda_1\lambda_2\big)\,t^2}{x_1x_2}, \\ Y &= \frac{8(x_2-1)(x_2-\lambda_2)\big(\lambda_2(\lambda_1+1)x_1+\lambda_1(\lambda_2+1)x_2-\lambda_1\lambda_2\big)\,t^2}{x_1^2x_2} \\ &\qquad + \frac{4\lambda_1\big((\lambda_1+1)x_1-2\lambda_1\big)\,t^4}{x_1} + \frac{4\lambda_2\big((\lambda_2+1)x_2-2\lambda_2\big)\,t^2}{x_2}. \end{aligned}$$

The discriminant is of the form $u^8d(u)$, where $d(u)$ is an irreducible polynomial of degree 8. Besides two IV^* fibers, the elliptic fibration has eight I_1 fibers in the generic case. These eight I_1 fibers can degenerate in four different ways; 2 I_2 + 4 I_1, 4 I_2, 4 II or 2 IV. For more detail, see Prop. 5.1 in [**14**].

There are eight other A_{ij}'s which define sections; the correspondence between these curves and the sections is as follows:

$$\begin{aligned} A_{12} &\leftrightarrow P_1 = \big(4u^2(\lambda_1^2u^2-\lambda_2(\lambda_1+1)), \\ &\qquad\qquad -4u^2(2\lambda_1^3u^4-\lambda_1(\lambda_1+1)(2\lambda_2-1)u^2+\lambda_2(\lambda_2-1))\big) \\ A_{13} &\leftrightarrow P_2 = \bigg(\frac{4u^2(\lambda_1^2u^2-\lambda_2^2(\lambda_1+1))}{\lambda_2^2}, \\ &\qquad\qquad -\frac{4u^2(2\lambda_1^3u^4+\lambda_1\lambda_2^2(\lambda_1+1)(\lambda_2-2)u^2-\lambda_2^4(\lambda_2-1))}{\lambda_2^3}\bigg) \\ A_{21} &\leftrightarrow P_3 = \big(-4(\lambda_1(\lambda_2+1)u^2-\lambda_2^2), \\ &\qquad\qquad -4(\lambda_1(\lambda_1-1)u^4-\lambda_2(\lambda_2+1)(2\lambda_1-1)u^2+2\lambda_2^3)\big) \\ A_{22} &\leftrightarrow P_4 = \big(-4\lambda_1\lambda_2u^2, -4u^2(\lambda_1(\lambda_1-1)u^2+\lambda_2(\lambda_2-1))\big) \\ A_{23} &\leftrightarrow P_5 = \big(-4\lambda_1u^2, -4u^2(\lambda_1(\lambda_1-1)u^2-\lambda_2(\lambda_2-1))\big) \\ A_{31} &\leftrightarrow P_6 = \bigg(-\frac{4(\lambda_1^2(\lambda_2+1)u^2-\lambda_2^2)}{\lambda_1^2}, \\ &\qquad\qquad \frac{4(\lambda_1^4(\lambda_1-1)u^4-\lambda_1^2\lambda_2(\lambda_1-2)(\lambda_2+1)u^2-2\lambda_2^3)}{\lambda_1^3}\bigg) \\ A_{32} &\leftrightarrow P_7 = \big(-4\lambda_2u^2, 4u^2(\lambda_1(\lambda_1-1)u^2-\lambda_2(\lambda_2-1))\big) \\ A_{33} &\leftrightarrow P_8 = \big(-4u^2, 4u^2(\lambda_1(\lambda_1-1)u^2+\lambda_2(\lambda_2-1))\big) \end{aligned}$$

These sections satisfy the following relations:

$$P_1 = P_5 + P_8, \qquad P_2 = P_4 + P_7,$$
$$P_3 = P_7 + P_8, \qquad P_6 = P_4 + P_5.$$

We can show that P_4, P_8, P_5, and P_7 generate the Mordell-Weil group in the generic case. The height matrix with respect to the basis $\{P_4, P_8, P_5, P_7\}$ is

$$\begin{pmatrix} \frac{4}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{4}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & \frac{4}{3} \end{pmatrix}.$$

This is the direct sum of two copies of $A_2^*[2]$, the dual lattice of A_2 scaled by 2.

2.3. $\mathscr{J}_4$**.** The elliptic parameter for the fibration π_1 in Example 1.1 is given by

$$u = x_1,$$

while the elliptic parameter for π_2 is given by $u = x_2$. For π_1, the change of variables

$$X = u(u-1)(u-\lambda_1)x_2,$$
$$Y = u^2(u-1)^2(u-\lambda_1)^2\, t,$$

converts the equation (1.1) to

$$Y^2 = X\big(X - u(u-1)(u-\lambda_1)\big)\big(X - \lambda_2\, u(u-1)(u-\lambda_1)\big).$$

The curve G_0 is the 0-section. Other sections are:

$$\begin{aligned} G_1 &\quad\leftrightarrow\quad T_1 = (0,0), \\ G_2 &\quad\leftrightarrow\quad T_2 = (u(u-1)(u-\lambda_1), 0), \\ G_3 &\quad\leftrightarrow\quad T_3 = (\lambda_2\, u(u-1)(u-\lambda_1), 0). \end{aligned}$$

Similar results hold for π_2.

2.4. $\mathscr{J}_6$**.** The divisor of the function x_1/x_2 is given by

$$\begin{aligned} \left(\frac{x_1}{x_2}\right) &= 2(F_1 + A_{10} + G_0) + A_{12} + A_{13} + A_{20} + A_{30} \\ &\qquad - \big(2(F_0 + A_{01} + G_1) + A_{02} + A_{03} + A_{21} + A_{31}\big). \end{aligned}$$

This is the difference of two disjoint divisors of type I_2^*, and thus

$$u = \frac{x_1}{x_2}$$

is an elliptic parameter of type $\mathscr{J}_6$.

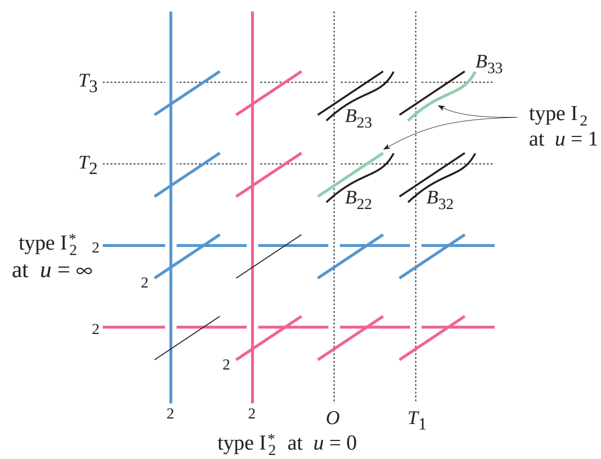

FIGURE 5. $\mathscr{J}_6$

In order to write down a Weierstrass equation using the curve F_2 as the 0-section, we put

$$X = \frac{x_1(x_1-\lambda_1)(x_1-x_2)(\lambda_2 x_1 - x_2)}{(x_1-1)x_2^3},$$
$$Y = \frac{(\lambda_1-1)\,t\,x_1^3(x_1-\lambda_1)(x_1-x_2)(\lambda_2 x_1 - x_2)}{(x_1-1)x_2^5}.$$

Then we obtain the Weierstrass equation

$$Y^2 = X\big(X - u(u-1)(\lambda_2 u - \lambda_1)\big)\big(X - u(u-\lambda_1)(\lambda_2 u - 1)\big).$$

Its discriminant is given by

$$\Delta(u) = 16u^8(\lambda_1-1)^2(\lambda_2-1)^2(u-1)^2(u-\lambda_1)^2(\lambda_2 u-1)^2(\lambda_2 u-\lambda_1)^2.$$

Besides two I_2^* fibers at $u = 0$ and ∞, there are four I_2 fibers at $u = 1, \lambda_1, 1/\lambda_2$ and λ_1/λ_2. This elliptic surface has the following three 2-torsion sections:

$$\begin{aligned} F_3 \quad &\leftrightarrow \quad T_1 = (0,0), \\ G_2 \quad &\leftrightarrow \quad T_2 = \big(u(u-\lambda_1)(\lambda_2 u - 1), 0\big), \\ G_3 \quad &\leftrightarrow \quad T_3 = \big(u(u-1)(\lambda_2 u - \lambda_1), 0\big), \end{aligned}$$

Note that A_{22}, A_{23}, A_{32}, and A_{33} are components of four I_2 fibers. The other components of these four I_2 fibers are new (-2)-curves not among the basic curves, which will be clarified in §3.2.

3. More (-2)-curves

In order to describe elliptic parameters for other types, we need more (-2)-curves than the basic curves. When we constructed elliptic parameters of type $\mathscr{J}_6$ just above, we obtained some new (-2)-curves as components of I_2 fibers. In this section we give a systematic way to obtain such (-2)-curves.

For our purpose, it is convenient to regard $X = \mathrm{Km}(C_1 \times C_2)$ as a double cover of the product of projective lines: $\mathbf{P}^1 \times \mathbf{P}^1 = \{(x_1 : z_1), (x_2 : z_2)\}$. Let $p_i : C_i \to \mathbf{P}^1$ $(i = 1, 2)$ be the projection given by

$$\begin{array}{cccc} p_i : & C_i & \longrightarrow & \mathbf{P}^1 \\ & (x_i : y_i : z_i) & \longmapsto & \begin{cases} (x_i : z_i) & \text{if } z_i \neq 0 \\ (1 : 0) & \text{if } z_i = 0 \end{cases} \end{array}$$

Then the map $p_1 \times p_2 : A = C_1 \times C_2 \to \mathbf{P}^1 \times \mathbf{P}^1$ factors through $\bar{\pi} : A/\iota_A \to \mathbf{P}^1 \times \mathbf{P}^1$. Let π be the morphism of degree two from X to $\mathbf{P}^1 \times \mathbf{P}^1$ that makes the following diagram commutative:

$$\begin{tikzcd} & X \arrow[d] \arrow[dr, "\pi"] & \\ A \arrow[r] & A/\iota_A \arrow[r, "\bar{\pi}"'] & \mathbf{P}^1 \times \mathbf{P}^1 \end{tikzcd}$$

We denote by R_{ij} the point in $\mathbf{P}^1 \times \mathbf{P}^1$ that is the image of the exceptional curve A_{ij} by π. To obtain more (-2)-curves, we look for curves in $\mathbf{P}^1 \times \mathbf{P}^1$ which lift to a (-2)-curve via the map π.

3.1. $(1,1)$**-curves.** Let L be a curve in $\mathbf{P}^1 \times \mathbf{P}^1$ defined by a bihomogeneous equation of bidegree $(1, 1)$:

$$ax_1x_2 + bx_1z_2 + cz_1x_2 + dz_1z_2 = 0.$$

We call such a curve $(1, 1)$-curve for short. By an abuse of notation, we denote the image of F_i and G_i under $\pi : S \to \mathbf{P}^1 \times \mathbf{P}^1$ by the same letters F_i and G_i, respectively. For example, F_1 is the curve with the equation $x_1 = 0$, and G_3 with $x_2 - \lambda_2 z_2 = 0$, etc.

Let L be a $(1, 1)$-curve in $\mathbf{P}^1 \times \mathbf{P}^1$. Its pullback $\pi^{-1}(L)$ ramifies at the intersection of L and F_i or G_j, except when the intersection point falls on $R_{ij} = F_i \cap G_j$.

Lemma 4. *Let L be a $(1, 1)$-curve. Then,*

1. *If L passes three of sixteen R_{ij}'s, then $\pi^{-1}(L)$ is a curve of genus* 0.
2. *If L passes two out of sixteen R_{ij}'s, then $\pi^{-1}(L)$ is a curve of genus* 1.

PROOF. In general a $(1,1)$-curve L intersects with $\sum F_i$ (resp. $\sum G_j$) at four points. If L passes three of sixteen R_{ij}'s, then it intersects with F_i one more time and G_j one more time outside R_{ij}. This implies that $\pi^{-1}(L)$ ramifies at two points. By Hurwitz's theorem $\pi^{-1}(L)$ is a curve of genus 0. Similarly, if L passes two out of sixteen R_{ij}'s, $\pi^{-1}(L)$ ramifies at four points, and it is a curve of genus 1. □

A $(1,1)$-curve is uniquely determined by a set of three points in a general position. If we choose $R_{i_0j_0}$, $R_{i_1j_1}$, $R_{i_2j_2}$ so that no two of them are on the same F_i or G_j, then they are in general position. Let i_3 and j_3 be the missing indices. In other words, we choose i_3 and j_3 such that $\{i_0, i_1, i_2, i_3\} = \{j_0, j_1, j_2, j_3\} = \{0, 1, 2, 3\}$. Under the condition that the two elliptic curves C_1 and C_2 are not isomorphic, the $(1,1)$-curve passing through $R_{i_0j_0}$, $R_{i_1j_1}$, and $R_{i_2j_2}$ does not pass $R_{i_3j_3}$. Thus, choosing $R_{i_0j_0}$, $R_{i_1j_1}$, $R_{i_2j_2}$ we obtain a $(1,1)$-curve whose pullback by π is an irreducible (-2)-curve in X. We denote such a $(1,1)$-curve by $L_{i_0j_0,i_1j_1,i_2j_2}$, and its pullback by $\tilde{L}_{i_0j_0,i_1j_1,i_2j_2}$. There are ninety-six such (-2)-curves. Also note that $\tilde{L}_{i_0j_0,i_1j_1,i_2j_2}$ intersects twice with each of $A_{i_0j_0}$, $A_{i_1j_1}$, and $A_{i_2j_2}$.

The $(1,1)$-curve $L_{00,11,22}$ passes through R_{00}, R_{11}, R_{22}. It is given by the bihomogeneous equation $x_2z_1 - x_1z_2 = 0$. In the sequel we write it in the affine form $x_2 - x_1 = 0$ for simplicity. $\tilde{L}_{00,11,22}$ is denoted by A^{44} in Oguiso [**8**], which appears in the $\mathscr{J}_2$ fibration. We denote it by B_{33} to make it consistent with our notation, indicating that it intersects with F_3 and G_3 outside A_{33}. Note, however, that there are six (-2)-curves of the form $\tilde{L}_{i_0j_0,i_1j_1,i_2j_2}$ that intersect with F_3 and G_3.

Fig. 6 shows the curve B_{33} in the affine space $\mathbf{A}_{x_1} \times \mathbf{A}_{x_2} \times \mathbf{A}_t$. As a matter of fact, if we substitute x_2 by x_1 in (1.1), the equation factorizes into

$$x_1(x_1 - 1)(x_1t^2 - x_1 - t^2\lambda_1 + \lambda_2) = 0,$$

which implies that the intersection between $x_2 - x_1 = 0$ and the affine Kummer surface (1.1) has three irreducible components, namely A_{11}, A_{22}, and B_{33}. We also see that a parametrization of B_{33} is given by

$$(x_1, x_2, t) = \left(\frac{\lambda_1 s^2 - 1}{s^2 - 1}, \frac{\lambda_1 s^2 - 1}{s^2 - 1}, s\right).$$

The zero divisor of the function $x_2 - x_1 \in k(x_1, x_2, t)$ is $A_{11} + A_{22} + B_{33}$, while the polar divisor is of the form $D_1 + D_2 + rA_{00}$, where

$$\begin{aligned} D_1 &= 2F_0 + A_{00} + A_{01} + A_{02} + A_{03}, \\ D_2 &= 2G_0 + A_{00} + A_{10} + A_{20} + A_{30}. \end{aligned}$$

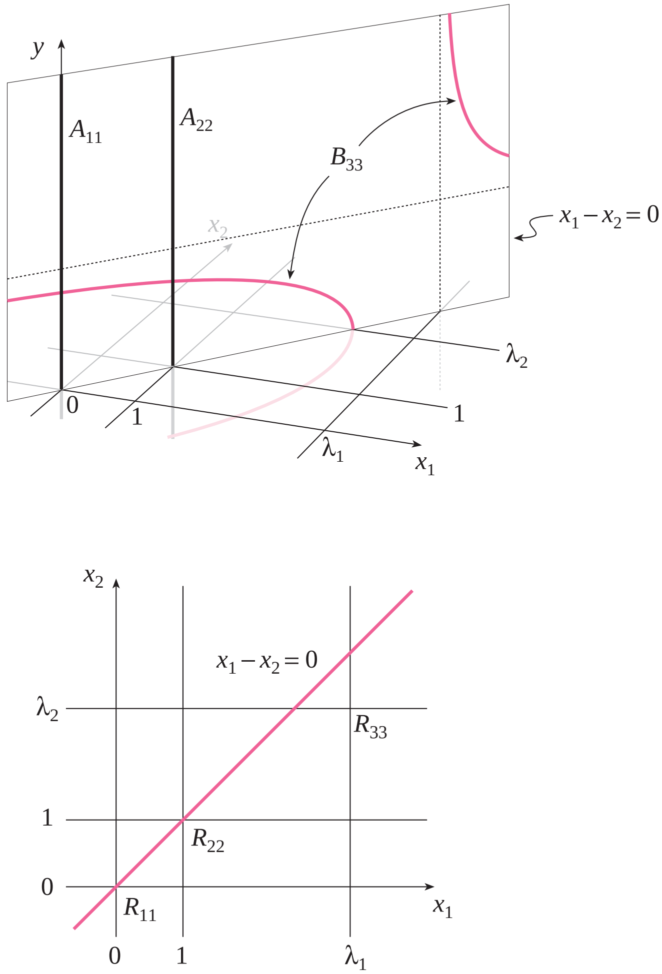

FIGURE 6. (-2)-curve B_{33}

Since A_{00} intersects twice with the divisor $A_{11}+A_{22}+B_{33}$, the intersection number $A_{00}\cdot(D_1+D_2+rA_{00})$ must be 2, which implies $r=-1$. This shows

$$
\begin{aligned}
(x_2-x_1) &= A_{11}+A_{22}+B_{33} \\
&\qquad -(2F_0+2G_0+A_{00}+A_{01}+A_{02}+A_{03}+A_{10}+A_{20}+A_{30}).
\end{aligned}
$$

This and similar calculations of divisors are used to find the elliptic parameter with a prescribed divisor in §4 and §5.

3.2. I_2 fibers of type $\mathscr{J}_6$ fibration. The elliptic parameter $u = x_1/x_2$, which is of type $\mathscr{J}_6$, defines a pencil of $(1,1)$-curves $x_1 - ux_2 = 0$.

The general fiber of this elliptic fibration is the pullback of a $(1,1)$-curve passing through R_{00} and R_{11}. If $x_1 - ux_2 = 0$ passes through a third R_{ij}, then its pullback is a singular fiber (see Fig. 7). Four fibers of type I_2, which

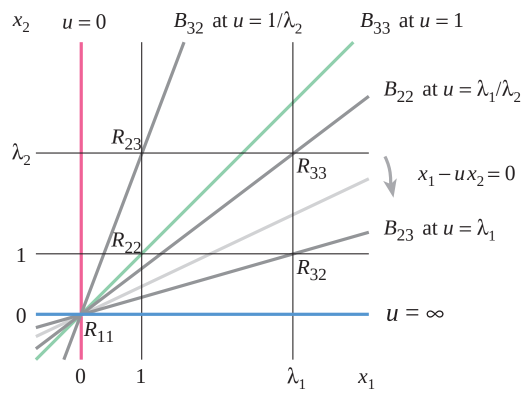

FIGURE 7. pencil of $(1,1)$-curves

are mentioned in §2.4 arise as follows:

$$\begin{array}{llll} \tilde{L}_{00,11,23} + A_{23} & \text{at } u = 1/\lambda_2, & B_{33} + A_{22} & \text{at } u = 1, \\ \tilde{L}_{00,11,33} + A_{33} & \text{at } u = \lambda_1/\lambda_2, & \tilde{L}_{00,11,32} + A_{32} & \text{at } u = \lambda_1. \end{array}$$

3.3. Notation. Even though the notation "B_{33}" is ambiguous as we mentioned earlier, it is quite convenient. We thus use the following notation in the sequel:

(3.1)
$$\begin{array}{ll} B_{32} = \tilde{L}_{00,11,23} : \lambda_2 x_1 - x_2 = 0, & B_{33} = \tilde{L}_{00,11,22} : x_1 - x_2 = 0, \\ B_{22} = \tilde{L}_{00,11,33} : \lambda_2 x_1 - \lambda_1 x_2 = 0, & B_{23} = \tilde{L}_{00,11,32} : x_1 - \lambda_1 x_2 = 0,. \end{array}$$

Later in §4.3 and §5.2, we introduce more (-2)-curves of this type, B_{31}, B_{12} and B_{13}.

4. Elliptic parameters for $\mathscr{J}_2$, $\mathscr{J}_7$, $\mathscr{J}_8$ and $\mathscr{J}_{11}$

4.1. $\mathscr{J}_2$. Using B_{33}, we can construct an elliptic parameter of type $\mathscr{J}_2$. In fact, the divisor

$$\Psi_{2,0} = F_3 + A_{33} + G_3 + B_{33}$$

is a divisor of type I_4, and it does not intersect with the divisor of type I_{12} given by

$$\Psi_{2,\infty} = F_0 + A_{02} + G_2 + A_{12} + F_1 + A_{10} \\ + G_0 + A_{20} + F_2 + A_{21} + G_1 + A_{01}$$

(see Fig. 8 below). It turns out that the divisor of the function

$$u = \frac{t(x_1 - \lambda_1)(x_1 - x_2)}{x_2(x_2 - 1)}$$

is $\Psi_{2,0} - \Psi_{2,\infty}$, and it is an elliptic parameter of type $\mathscr{J}_2$. Choosing A_{30} as

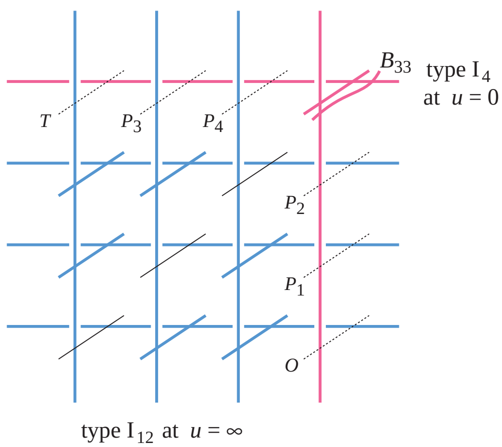

FIGURE 8. $\mathscr{J}_2$

the 0-section, we obtain the Weierstrass equation

$$\begin{aligned} Y^2 = X^3 + \big(u^4 + 2\,(2\lambda_1\lambda_2 - \lambda_1 - \lambda_2 + 2)\,u^2 + (\lambda_2 - \lambda_1)^2\big)X^2 \\ - 16\lambda_1\lambda_2(\lambda_1 - 1)(\lambda_2 - 1)\,u^2 X, \end{aligned}$$

where the change of variables is given by

$$\begin{aligned} X &= -\frac{4\lambda_1(\lambda_1 - 1)(x_1 - x_2)(x_2 - \lambda_2)}{x_1(x_1 - 1)}, \\ Y &= -\frac{4\lambda_1(\lambda_1 - 1)(x_1 - x_2)(x_2 - \lambda_2)(2x_1 - 2x_2 - \lambda_1 + \lambda_2)}{x_1(x_1 - 1)} \\ &\qquad + \frac{4\lambda_1(\lambda_1 - 1)(x_1 - x_2)^2(x_2 - \lambda_2)^3(2x_1x_2 - x_1 - x_2)}{t^2 x_1^3(x_1 - 1)^3}. \end{aligned}$$

The discriminant of the fibration is of the form $u^4 d(u)$, where $d(u)$ is a polynomial of degree 8. The discriminant of $d(u)$ vanishes if and only if

$$\lambda_2 = \lambda_1, 1 - \lambda_1, \frac{1}{\lambda_1}, \text{or } \frac{\lambda_1}{\lambda_1 - 1}.$$

The curve A_{03} corresponds to the 2-torsion section $T = (0,0)$. The correspondence between the curves and the sections is as follows:

$$\begin{aligned}
A_{31} &\leftrightarrow P_1 = \big(4\lambda_1\lambda_2, 4\lambda_1\lambda_2(u^2+\lambda_1+\lambda_2)\big)\big) \\
A_{32} &\leftrightarrow P_2 = \big(4(\lambda_1-1)(\lambda_2-1), \\
&\qquad -4(\lambda_1-1)(\lambda_2-1)(u^2-\lambda_1-\lambda_2+2)\big)\big) \\
A_{13} &\leftrightarrow P_3 = \big(-4u^2(\lambda_1-1)(\lambda_2-1), \\
&\qquad 4(\lambda_1-1)(\lambda_2-1)u^2(u^2+\lambda_1+\lambda_2)\big)\big) \\
A_{23} &\leftrightarrow P_4 = \big(-4\lambda_1\lambda_2u^2, -4\lambda_1\lambda_2u^2(u^2-\lambda_1-\lambda_2+2)\big)\big)
\end{aligned}$$

These sections satisfy the following relations.

$$P_3 = P_1 + T, \qquad P_4 = P_2 + T.$$

The Mordell-Weil group is generated by T, P_1 and P_2 in the general case where C_1 and C_2 are not isogenous. The height matrix with respect to $\{P_1, P_2\}$ is shown to be

$$\begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}.$$

Thus the Mordell-Weil lattice is isomorphic to $A_2^*[2]$.

4.2. $\mathscr{J}_7$**.** Using the curves B_{33} and B_{32} introduced in §3.3, we can form two disjoint divisors of type I_0^*:

$$\begin{aligned}
\Psi_{7,0} &= 2G_3 + A_{03} + A_{13} + A_{33} + B_{33}, \\
\Psi_{7,\infty} &= 2G_2 + A_{02} + A_{12} + A_{32} + B_{32}.
\end{aligned}$$

Looking for the function whose divisor is $\Psi_{7,0} - \Psi_{7,\infty}$, we obtain the elliptic parameter

$$u = \frac{(x_2-\lambda_2)(x_1-x_2)}{(x_2-1)(\lambda_2x_1-x_2)}. \tag{4.1}$$

The divisor of the function

$$u - 1 = -\frac{(\lambda_2-1)\,x_2(x_1-1)}{(x_2-1)(\lambda_2x_1-x_2)}$$

is given by the following divisor consisting only of the basic curves:

$$A_{01} + A_{31} + 2G_1 + 2A_{21} + 2F_2 + 2A_{20} + 2G_0 + A_{10} + A_{30}.$$

This is a singular fiber of type I_4^* (see Fig. 9). Thus, the elliptic parameter given by (4.1) is of type $\mathscr{J}_7$.

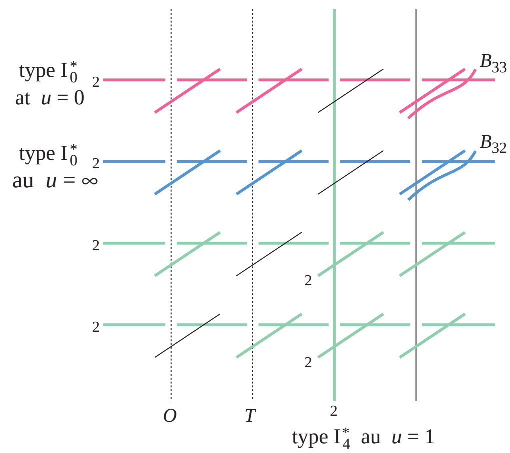

FIGURE 9. $\mathscr{J}_7$

The change of variables

$$\begin{aligned}
X &= \frac{\lambda_2 u(u-1)^2 x_1}{x_2} \\
&= \frac{\lambda_2(\lambda_2-1)^2 x_1(x_1-1)^2 x_2(x_2-\lambda_2)(x_1-x_2)}{(x_2-1)^3(\lambda_2 x_1-x_2)^3}, \\
Y &= \frac{\lambda_2(\lambda_2-1)u^2(u-1)^2}{t} \\
&= \frac{\lambda_2(\lambda_2-1)^3(x_1-1)^2 x_2^2(x_2-\lambda_2)^2(x_1-x_2)^2}{t(x_2-1)^4(\lambda_2 x_1-x_2)^4},
\end{aligned}$$

converts (1.1) to the Weierstrass equation

$$Y^2 = X^3 - u(u-1)\big((\lambda_1\lambda_2+1)u-\lambda_1-\lambda_2\big)X^2 + \lambda_1\lambda_2 u^2(u-1)^4 X.$$

Its discriminant is of the form $u^6(u-1)^{10}d(u)$, where $d(u)$ is a polynomial of degree 2.

Generically, it has only one section other than 0-section:

$$F_2 \quad \leftrightarrow \quad T=(0,0).$$

4.3. $\mathscr{J}_8$**.** To find an elliptic parameter of type $\mathscr{J}_8$, we need to construct a I_2^* fiber. For this, we can make use of B_{33} once again. The divisor

$$\Psi_{8,0} = A_{01}+A_{02}+2F_0+2A_{03}+2G_3+A_{33}+B_{33}$$

is of type I_2^* and it does not intersect with the divisor

$$\Psi_{8,\infty} = A_{12}+2F_1+3A_{10}+4E_0+3A_{20}+2F_2+A_{21}+2A_{30},$$

which is of type III*. We look for a function whose divisor is $\Psi_{8,0} - \Psi_{8,\infty}$, and we obtain the elliptic parameter of type $\mathscr{J}_8$

$$u = -\frac{(x_2 - \lambda_2)(x_1 - x_2)}{\lambda_2(\lambda_2 - 1)x_1(x_1 - 1)}.$$

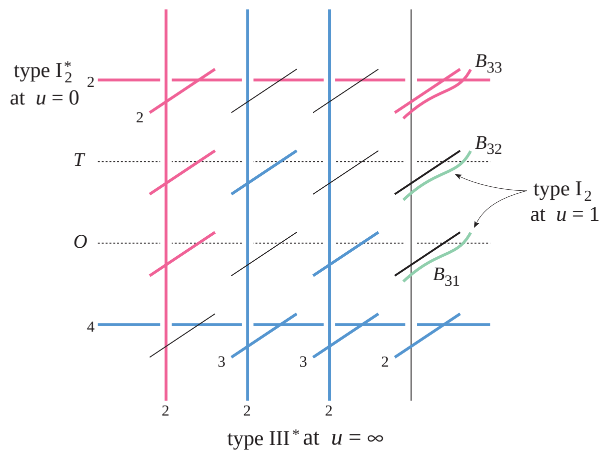

FIGURE 10. $\mathscr{J}_8$

Let B_{31} be the (-2)-curve $\tilde{L}_{00,13,22} : (\lambda_2 - 1)x_1 + x_2 - \lambda_2 = 0$. Then B_{32} and B_{31} form a fiber of type I_2 at $u = 1$. Also A_{32} and the pullback of a certain $(2,2)$-curve form another fiber of type I_2 at $u = 1/(\lambda_1\lambda_2)$, while A_{31} together with the pullback of a certain $(2,2)$-curve form the third fiber of type I_2 at $u = (\lambda_1 - 1)^{-1}(\lambda_2 - 1)^{-1}$. The change of variables

$$X = u\big((\lambda_1 - 1)(\lambda_2 - 1)u - 1\big)\frac{(x_2 - 1)(\lambda_2 x_1 - x_2)}{(\lambda_2 - 1)x_2(x_1 - 1)},$$
$$Y = -u^3\big((\lambda_1 - 1)(\lambda_2 - 1)u - 1\big)\frac{\lambda_2(x_2 - 1)(\lambda_2 x_1 - x_2)}{t\,x_2(x_1 - 1)},$$

converts (1.1) to the Weierstrass equation

$$Y^2 = X^3 - u\big((2\lambda_1\lambda_2 - \lambda_1 - \lambda_2 + 2)u - 2\big)X^2 - u^2(u - 1)(\lambda_1\lambda_2 u - 1)\big((\lambda_1 - 1)(\lambda_2 - 1)u - 1\big)X.$$

Its discriminant is

$$\Delta(u) = 16u^8(u - 1)^2(\lambda_1\lambda_2 u - 1)^2\big((\lambda_1 - 1)(\lambda_2 - 1)u - 1\big)^2 \times \big(4\lambda_1\lambda_2(\lambda_1 - 1)(\lambda_2 - 1)u + (\lambda_1 - \lambda_2)^2\big).$$

[If $\lambda_2 = -\lambda_1, 2-\lambda_2$, or $\lambda_1/(2\lambda_1 - 1)$, this elliptic fibration has fiber of type III for general λ_1.]

Generically, it has only one section other than 0-section:

$$G_2 \quad \leftrightarrow \quad T = (0,0).$$

4.4. $\mathscr{J}_{11}$**.** Modifying the divisors appearing in the type $\mathscr{J}_7$ fibration we constructed in §4.2, we form two divisors

$$\Psi_{11,0} = A_{31} + A_{21} + 2G_1 + 2A_{01} + 2F_0 + 2A_{03} + 2G_3 + A_{33} + B_{33},$$
$$\Psi_{11,\infty} = A_{30} + A_{20} + 2G_0 + 2A_{10} + 2F_1 + 2A_{12} + 2G_2 + A_{32} + B_{32}.$$

They are of type I_4^* and they do not intersect with each other.

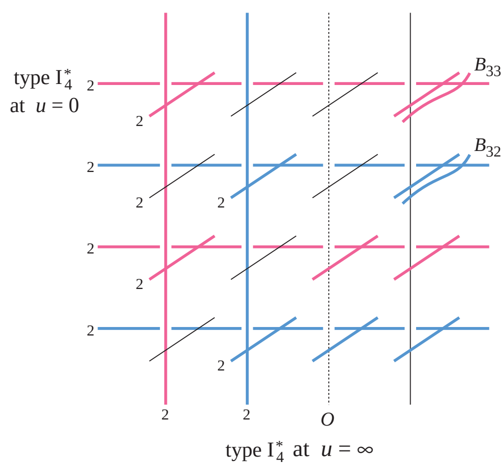

FIGURE 11. $\mathscr{J}_{11}$

We look for a function whose divisor is $\Psi_{11,0} - \Psi_{11,\infty}$, and we obtain the elliptic parameter of type $\mathscr{J}_{11}$:

$$u = \frac{x_2(x_2-\lambda_2)(x_1-x_2)}{x_1(x_2-1)(\lambda_2 x_1 - x_2)}.$$

The change of variables

$$\begin{aligned} X &= u\frac{(\lambda_1-1)(x_2-\lambda_2)(x_1-x_2)}{x_1(x_1-1)} \\ &= \frac{(\lambda_1-1)\,x_2(x_2-\lambda_2)^2(x_1-x_2)^2}{x_1^2(x_1-1)(x_2-1)(\lambda_2x_1-x_2)}, \\ Y &= u^2\frac{(\lambda_1-1)(x_2-\lambda_2)^2(x_1-x_2)^2}{t\,x_1^2(x_1-1)^2} \\ &= \frac{(\lambda_1-1)\,x_2^2(x_2-\lambda_2)^4(x_1-x_2)^4}{t\,x_1^4(x_1-1)^2(x_2-1)^2(\lambda_2x_1-x_2)^2} \end{aligned}$$

converts (1.1) to the Weierstrass equation

$$\begin{aligned} Y^2 &= X^3 + \bigl(\lambda_1u^2 - (2\lambda_1\lambda_2-\lambda_1-\lambda_2+2)u+\lambda_2\bigr)\,u\,X^2 \\ &+ (\lambda_1-1)(\lambda_2-1)\bigl((\lambda_1\lambda_2+1)u-2\lambda_2\bigr)\,u^3\,X + \lambda_2(\lambda_1-1)^2(\lambda_2-1)^2\,u^5. \end{aligned}$$

Its discriminant is of the form $u^{10}d(u)$, where $d(u)$ is a polynomial of degree 4. The discriminant of $d(u)$ is too complicated to write down here. However, a simple search reveals that there are cases where four I_1 fibers degenerate even when C_1 and C_2 are not isogenous.

Remark. Suppose that the characteristic of the base field is 0.

(1) If $\lambda_1 = -1$ and $\lambda_2 = 9 \pm 4\sqrt{5}$, then the fibration has one I_2 fiber and one type II fiber. In this case j-invariant of C_1 is 1728 and that of C_2 is $78608 = 2^4 17^3$. They are not isogenous, and they can be defined over $\mathbf{Q}$.

(2) If $\lambda_1 = -1$ and $\lambda_2 = \pm\sqrt{-1}$, then the fibration has two type II fibers. In this case j-invariant of C_1 is 1728 and that of C_2 is 128. They are not isogenous, and they can be defined over $\mathbf{Q}$.

5. $(2,2)$-curves and $\mathscr{J}_5$, $\mathscr{J}_9$ and $\mathscr{J}_{10}$

5.1. $(2,2)$**-curves.** Now the pullbacks of $(1,1)$-curves are not enough to construct all the elliptic fibrations in Oguiso's list. A pullback of a $(2,2)$-curve is a candidate for missing (-2)-curves. A nonsingular $(2,2)$-curve in $\mathbf{P}^1\times\mathbf{P}^1$ is a curve of genus 1, and thus, we first look for $(2,2)$-curves with a node. Then we try to impose conditions such that their pullbacks are (-2)-curves. Here, we do not try to make a systematic search as before.

Actually, we can construct an elliptic fibration of type $\mathscr{J}_5$ using only pullbacks of $(1,1)$-curves and the basic curves. As a by-product, however, we obtain some new (-2)-curves which are pullbacks of $(2,2)$-curves. Such curves have a node at R_{11}. They are given by an equation of the form

$$a\,x_1^2z_2^2 + b\,x_2x_1z_2z_1 + c\,x_2^2z_1^2 + d\,x_1z_2^2z_1 + e\,x_2z_2z_1^2 + f\,z_2^2z_1^2 = 0.$$

The fact that it has a node at R_{11} corresponds to the fact that the equation does not have the terms $x_1^2x_2^2$, $x_1^2x_2z_2$, and $x_1z_1x_2^2$. In order to obtain such

a $(2,2)$-curve, we need to specify six points among R_{ij} $(1 \leq i, j \leq 4)$ such that no three among them are on the same F_i or G_j. We use such curves to construct an elliptic fibration of type $\mathscr{J}_9$ and $\mathscr{J}_{10}$.

5.2. $\mathscr{J}_5$**.** An elliptic fibration of type $\mathscr{J}_5$ has six I_2 fibers together with one I_6^* fiber. In order to write down an elliptic parameter for $\mathscr{J}_5$, we need to identify these six I_2 fibers.

Let B_{33} and B_{32} be the (-2)-curves introduced in §3.3. Consider two more (-2)-curves of this type:

$$B_{12} := \tilde{L}_{00,23,31} : \lambda_2(x_1 - \lambda_1) + (\lambda_1 - 1)x_2 = 0,$$
$$B_{13} := \tilde{L}_{00,22,31} : x_1 - \lambda_1 + (\lambda_1 - 1)x_2 = 0.$$

Looking at Fig. 12, we see that B_{33} and B_{12} intersect each other only at two points above the intersection of lines $x_1 - x_2 = 0$ and $\lambda_2(x_1 - \lambda_1) + (\lambda_1 - 1)x_2 = 0$. Thus, the divisor $B_{33} + B_{12}$ is a singular fiber of type I_2. Similarly, $B_{32} + B_{13}$ is another singular fiber of type I_2. Furthermore, $B_{33} + B_{12}$ and $B_{32} + B_{13}$ do not intersect each other since the image of these curves in $\mathbf{A}_{x_1} \times \mathbf{A}_{x_2}$ intersect only at R_{ij} (see Fig. 12 below).

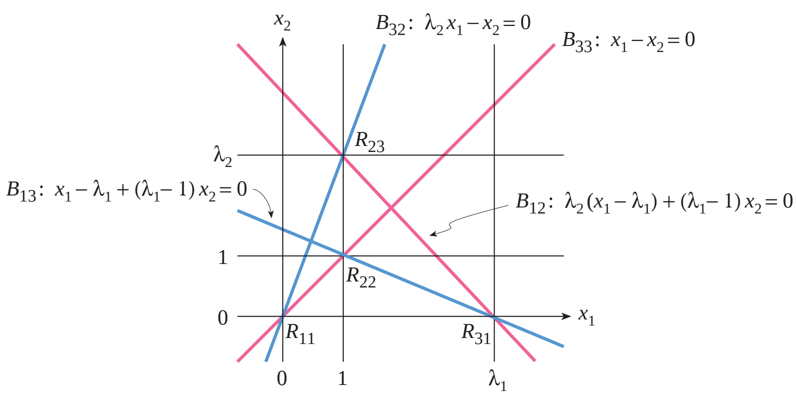

FIGURE 12. $(1,1)$-curves

Computing the divisors $(x_1 - x_2)$, $\big(\lambda_2(x_1 - \lambda_1) + (\lambda_1 - 1)x_2\big)$, $(\lambda_2 x_1 - x_2)$ and $\big(x_1 - \lambda_1 + (\lambda_1 - 1)x_2\big)$, we see that

$$u = \frac{(x_1 - x_2)\big(\lambda_2(x_1 - \lambda_1) + (\lambda_1 - 1)x_2\big)}{(\lambda_2 x_1 - x_2)\big(x_1 - \lambda_1 + (\lambda_1 - 1)x_2\big)}$$

is an elliptic parameter of type $\mathscr{J}_5$. We have

$$u - 1 = \frac{-\lambda_1(\lambda_2 - 1)\, x_2(x_1 - 1)}{(\lambda_2 x_1 - x_2)\big(x_1 - \lambda_1 + (\lambda_1 - 1)x_2\big)},$$

which shows that the fiber at $u = 1$ is a singular fiber of type I_6^*. Each of

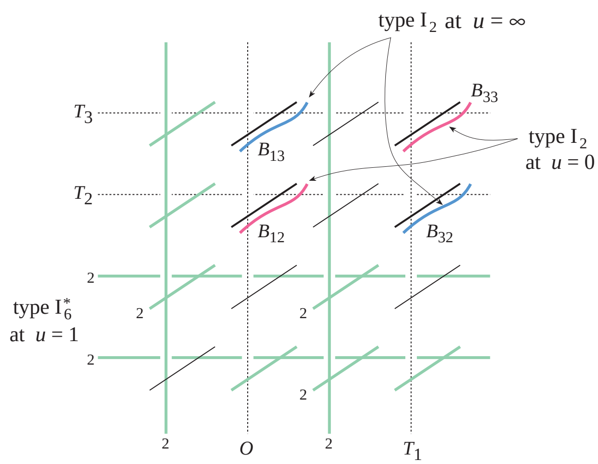

FIGURE 13. $\mathscr{J}_5$

the divisors A_{12}, A_{13}, A_{32} and A_{33} is a component of a singular fiber of type I_2. The other (-2)-curves are pullbacks of $(2,2)$-curves. For example, the singular fiber at $u = \lambda_1\lambda_2 - \lambda_1 + 1$ consists of A_{12} and the pullback of the $(2,2)$-curve given by

$$\lambda_2 x_1(x_1 - 1) + (\lambda_1 - 1)(x_2 - 1)(\lambda_2 x_1 - x_2) = 0.$$

In order to obtain a Weierstrass equation using the curve G_0 as the 0-section, we first put

$$X_0 = \frac{(x_1 - x_2)(x_1 - \lambda_1)}{x_1\big((x_1 - \lambda_1) + (\lambda_1 - 1)x_2\big)},$$
$$Y_0 = \frac{\lambda_1(\lambda_1 - 1)\, t\, x_2(x_1 - 1)(x_1 - \lambda_1)(x_2 - x_1)}{x_1(\lambda_2 x_1 - x_2)\big((x_1 - \lambda_1) + (\lambda_1 - 1)x_2\big)^2},$$

and then put

$$X = \lambda_1(\lambda_2 - 1)(u - 1)(u - \lambda_1\lambda_2 + \lambda_1 - 1)\big((\lambda_1\lambda_2 - \lambda_1 - \lambda_2)u + \lambda_2\big)X_0,$$
$$Y = \lambda_1^2(\lambda_2 - 1)^2(u - 1)^2(u - \lambda_1\lambda_2 + \lambda_1 - 1)\big((\lambda_1\lambda_2 - \lambda_1 - \lambda_2)u + \lambda_2\big)Y_0.$$

Then (X, Y) satisfy the Weierstrass equation

$$Y^2 = X(X - \alpha)(X - \beta),$$

where

$$\alpha = -\lambda_1(\lambda_2 - 1)(u - 1)\big((\lambda_1\lambda_2 - 1)u - \lambda_1 + 1\big)\big((\lambda_1\lambda_2 - \lambda_1 - \lambda_2)u + \lambda_2\big),$$
$$\beta = \lambda_1(\lambda_2 - 1)u(u - 1)(u - \lambda_1\lambda_2 + \lambda_1 - 1)\big((\lambda_1\lambda_2 - \lambda_2)u - \lambda_1 + \lambda_2\big).$$

The discriminant of this fibration is given by

$$\begin{aligned}\Delta(u) = 16\lambda_1^6\lambda_2^2(\lambda_1-1)^2u^2(u-1)^{12}\\ \times (u-\lambda_1\lambda_2+\lambda_1-1)^2\big((\lambda_1\lambda_2-1)u-\lambda_1+1\big)^2\\ \times \big((\lambda_1\lambda_2-\lambda_2)u-\lambda_1+\lambda_2\big)^2\big((\lambda_1\lambda_2-\lambda_1-\lambda_2)u+\lambda_2\big)^2.\end{aligned}$$

The Mordell-Weil group of this elliptic surface has the following three sections:

$$\begin{aligned} F_3 \quad &\leftrightarrow \quad T_1=(0,0),\\ G_2 \quad &\leftrightarrow \quad T_2=(\alpha,0),\\ G_3 \quad &\leftrightarrow \quad T_3=(\beta,0).\end{aligned}$$

5.3. $\mathscr{J}_9$**.** In order to construct an elliptic fibration of type $\mathscr{J}_9$, we need to find a divisor of type I_0^* different from the ones appearing in $\mathscr{J}_4$ or $\mathscr{J}_7$. To do so we look for a (-2)-curve P_{33} such that $2G_3+A_{03}+A_{33}+B_{33}+P_{33}$ is of type I_0^*. We can show that P_{33} cannot be a pullback of a $(1,1)$-curve; if that were the case, B_{33} and P_{33} would have to intersect each other. Thus, we look for a $(2,2)$-curve whose double cover serves as P_{33}.

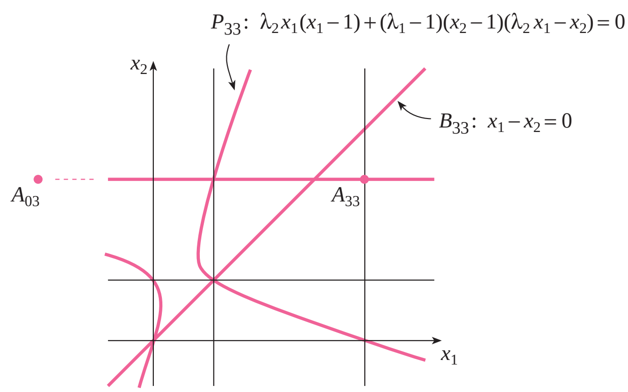

FIGURE 14. fiber at $u=0$

It turns out that the pullback of the $(2,2)$-curve

$$\lambda_2x_1(x_1-1)+(\lambda_1-1)(x_2-1)(\lambda_2x_1-x_2)=0 \tag{5.1}$$

can be used as P_{33}. This curve is a component of a I_2 fiber of the elliptic fibration of type $\mathscr{J}_5$ which we constructed in the previous subsection. The $(2,2)$-curve (5.1) has a node at R_{00}, and passes through $R_{11}, R_{12}, R_{22}, R_{23}$, and R_{31}. Fig. 14 shows the projection of the (-2)-curves contained in the

divisor $\Psi_{9,0} = 2G_3 + A_{03} + A_{33} + B_{33} + P_{33}$. (The projection of A_{03} is R_{03}, which is a point at infinity.)

Similarly, let P_{32} be the pullback of the $(2,2)$-curve

$$\lambda_2 x_1(x_1 - 1) + (\lambda_1 - 1)(x_2 - \lambda_2)(x_1 - x_2) = 0.$$

Then, the divisor $\Psi_{9,\infty} = 2G_2 + A_{02} + A_{32} + B_{32} + P_{32}$ is again of type I_0^*, which does not intersect with $\Psi_{9,0}$. Fig. 15 shows the curves contained in the divisor $\Psi_{9,\infty}$. Looking for the function having the divisor $\Psi_{9,0} - \Psi_{9,\infty}$,

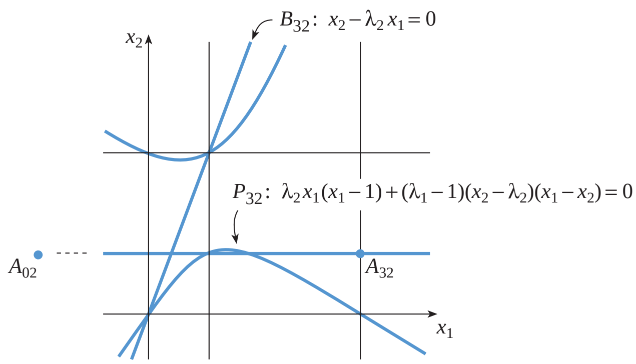

FIGURE 15. fiber at $u = \infty$

we find the elliptic parameter u given by

(5.2)
$$u = \frac{(x_2 - \lambda_2)(x_1 - x_2)\big(\lambda_2 x_1(x_1 - 1) + (\lambda_1 - 1)(x_2 - 1)(\lambda_2 x_1 - x_2)\big)}{(x_2 - 1)(\lambda_2 x_1 - x_2)\big(\lambda_2 x_1(x_1 - 1) + (\lambda_1 - 1)(x_2 - \lambda_2)(x_1 - x_2)\big)}.$$

We have

$$u - 1 = \frac{-\lambda_2(\lambda_2 - 1)x_1 x_2 (x_1 - 1)^2}{(x_2 - 1)(\lambda_2 x_1 - x_2)\big(\lambda_2 x_1(x_1 - 1) + (\lambda_1 - 1)(x_2 - \lambda_2)(x_1 - x_2)\big)}.$$

The zero divisor of this function $u - 1$ is given by the following divisor consisting only of the basic curves:

$$A_{01} + 2G_1 + 3A_{21} + 4F_2 + 5A_{20} + 6G_0 + 3A_{30} + 4A_{10} + 2F_1.$$

This is a singular fiber of type II^* (see Fig. 16). Thus, the elliptic parameter given by (5.2) is of type $\mathscr{J}_9$.

Our next task is to write down a Weierstrass equation. If we regard (5.2) as the defining equation of a curve in $\mathbf{P}^1 \times \mathbf{P}^1$ defined over $k(u)$, then we can

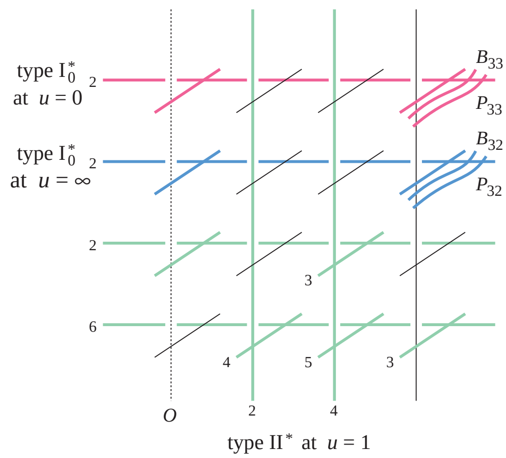

FIGURE 16. $\mathscr{J}_9$

show that this curve is a curve of genus 0, and thus it can be parametrized. In fact, we can parametrize x_1 and x_2 satisfying (5.2) using the parameter

$$\xi = -\frac{(x_2-1)(\lambda_2 x_1 - x_2)}{\lambda_2 x_1 x_2}.$$

Actual parametrizations of x_1 and x_2 are complicated and we omit here. Substituting x_1 and x_2 in the equation (1.1) by these parametrizations, we obtain an equation of a curve of genus 1 with variables in (ξ, t) defined over $k(u)$. This equation turns out to be a quadratic equation in t, and it is easily converted to a Weierstrass equation. Combining all these, we obtain the change of variables

$$X_0 = -\frac{\lambda_2(\lambda_2-1)x_1(x_1-1)}{(x_2-1)(\lambda_2 x_1 - x_2)}, \quad Y_0 = \frac{\lambda_2(\lambda_2-1)}{t}$$

that converts (1.1) to a twisted form of Weierstrass equation

$$\begin{aligned} u(u-1)Y_0^2 = X_0^3 &+ (\lambda_1\lambda_2 - 2\lambda_1 - 2\lambda_2 + 1)(u-1)X_0^2 \\ &- (\lambda_1 + \lambda_2 - 1)(\lambda_1\lambda_2 - \lambda_1 - \lambda_2)(u-1)^2 X_0 \\ &- \lambda_1\lambda_2(\lambda_1 - 1)(\lambda_2 - 1)(u-1)^2. \end{aligned}$$

By letting

$$X = u(u-1)X_0, \quad Y = u^2(u-1)^2 Y_0,$$

we obtain the following Weierstrass equation:

$$
\begin{aligned}
Y^2 = X^3 &+ (\lambda_1\lambda_2 - 2\lambda_1 - 2\lambda_2 + 1)\, u\,(u-1)^2 X^2 \\
&\qquad - (\lambda_1 + \lambda_2 - 1)(\lambda_1\lambda_2 - \lambda_1 - \lambda_2)\, u^2 (u-1)^4 X \\
&\qquad\qquad - \lambda_1\lambda_2(\lambda_1 - 1)(\lambda_2 - 1)\, u^3 (u-1)^5.
\end{aligned}
$$

Its discriminant is of the form $u^6(u-1)^{10}d(u)$, where $d(u)$ is a polynomial of degree 2. The discriminant of $d(u)$ is given by

$$16\lambda_1^2\lambda_2^2(\lambda_1-1)^2(\lambda_2-1)^2(\lambda_1^2-\lambda_1+1)^3(\lambda_2^2-\lambda_2+1)^3.$$

If either λ_1 or λ_2 is a sixth root of unity, then two I_1 fibers of the fibration degenerate to form a type II fiber.

5.4. $\mathscr{J}_{10}$**.** In order to construct an elliptic fibration of type $\mathscr{J}_{10}$, we must find yet another divisor of type I_0^*. The divisor $\Psi_{9,0} = 2G_3 + A_{03} + A_{33} + B_{33} + P_{33}$ is a divisor of type I_0^* appearing in the elliptic fibration constructed in the previous subsection. Since neither B_{33} nor P_{33} intersects with A_{13}, we see that

$$\Psi_{10,0} = 2G_3 + A_{13} + A_{33} + B_{33} + P_{33}$$

is also a divisor of type I_0^*. We then find a divisor of type I_6^* that does not intersect with $\Psi_{10,0}$:

$$
\begin{aligned}
\Psi_{10,\infty} = B_{32} &+ A_{32} + 2G_2 + 2A_{02} + 2F_0 + 2A_{01} \\
&\qquad + 2G_1 + 2A_{21} + 2F_2 + 2A_{20} + 2G_0 + A_{10} + A_{30}
\end{aligned}
$$

(see Fig.17). Looking for the function having the divisor $\Psi_{10,0} - \Psi_{10,\infty}$, we find the elliptic parameter of type $\mathscr{J}_{10}$ given by

$$u = \frac{(x_2-\lambda_2)(x_1-x_2)\big((\lambda_1-1)(x_2-1)(\lambda_2x_1-x_2)+\lambda_2x_1(x_1-1)\big)}{x_2(x_2-1)(x_1-1)(\lambda_2x_1-x_2)}. \tag{5.3}$$

The curve in $\mathbf{P}^1\times\mathbf{P}^1$ over $k(u)$ defined by (5.3) is a curve of genus 0. As in the case of $\mathscr{J}_9$, the parameter

$$\xi = \frac{(x_1-x_2)(x_2-\lambda_2)}{(x_1-1)x_2},$$

can be used to parametrize this curve. We can proceed in a similar manner to the case of $\mathscr{J}_9$ and we obtain the change of variables

$$
\begin{aligned}
X_0 &= \frac{\lambda_1(\lambda_1-1)(x_2-1)(\lambda_2x_1-x_2)}{x_1(x_1-1)}, \\
Y_0 &= \frac{\lambda_1(\lambda_1-1)(x_2-1)^2(\lambda_2x_1-x_2)^2}{tx_1^2(x_1-1)^2},
\end{aligned}
$$

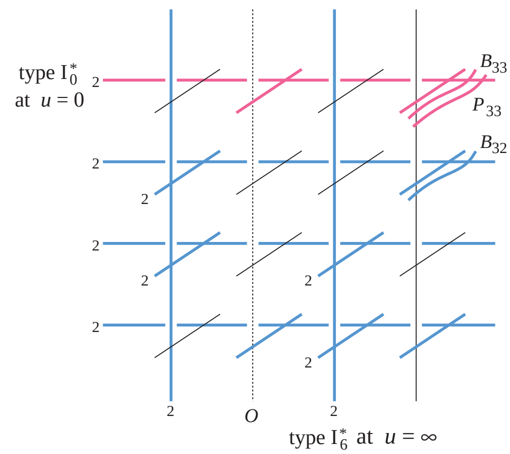

FIGURE 17. $\mathscr{J}_{10}$

which converts (1.1) to

$$\begin{aligned} uY_0^2 = X_0^3 &- (u + \lambda_1 + \lambda_2 - 1)(u - \lambda_1\lambda_2 + \lambda_1 + \lambda_2)X_0^2 \\ &+ \lambda_1\lambda_2(\lambda_1 - 1)(\lambda_2 - 1)(2u - \lambda_1\lambda_2 + 2\lambda_1 + 2\lambda_2 - 1)X_0 \\ &+ \lambda_1^2\lambda_2^2(\lambda_1 - 1)^2(\lambda_1 - 1)^2. \end{aligned}$$

Putting

$$X = uX_0, \quad Y = u^2 Y_0,$$

we obtain the Weierstrass equation

$$\begin{aligned} Y^2 = X^3 &+ u(u - \lambda_1 - \lambda_2 + 1)(u + \lambda_1\lambda_2 - \lambda_1 - \lambda_2)X^2 \\ &+ \lambda_1\lambda_2(\lambda_1 - 1)(\lambda_2 - 1)u^2(2u + \lambda_1\lambda_2 - 2\lambda_1 - 2\lambda_2 + 1)X \\ &+ \lambda_1^2\lambda_2^2(\lambda_1 - 1)^2(\lambda_1 - 1)^2 u^3. \end{aligned}$$

Its discriminant is of the form $u^6 d(u)$, where $d(u)$ is a polynomial of degree 2. We can show that $d(u)$ can have a multiple root without C_1 and C_2 being isogenous.

6. Full list of the defining equations in a special case

In this section, we take as C_1 and C_2 the most familiar elliptic curves

(6.1) $$C_1 : y_1^2 = x_1^3 - x_1, \quad C_2 : y_2^2 = x_2^3 - 1,$$

and write down the full list of the defining equations of mutually nonisomorphic elliptic fibrations on the Kummer surface $S = \mathrm{Km}(C_1 \times C_2)$ in characteristic 0. Although they are very special among elliptic curves (e.g.

automorphisms or complex multiplications), corrseponding Kummer surface S serves as a more or less "typical" case, since C_1, C_2 are not isogenous to each other.

In this case, the number $N(n)$ of nonisomorphic elliptic fibrations on S of type $\mathscr{J}_n$ has been determined by Oguiso as follows:

$$N(n) = 1 \quad \text{for } n = 2, 3, 5, 8, 9, 10,$$

and

$$N(n) = 2 \quad \text{for } n = 1, 4, 6, 7, 11.$$

(See Oguiso [**8**, p. 652]. We note that this number N is *not* typical among all non-isogenous curves, as shown there.)

Now observe that the values of Legendre parameter λ_i for the present C_i are as follows:

$$\lambda_1 = -1, -2 \text{ or } 1/2, \quad \lambda_2 = -\omega \text{ or } -\omega^2,$$

where ω is a cubic root of unity. In the following, we write down the $N = N(n)$ defining equations for each type $\mathscr{J}_n$. When $N = 1$, we give essentially the same equation as the one constructed in the previous sections, except that we make some coordinate change when it makes the equation look simpler. When $N > 1$, we make the same construction as before using a suitable equivalent value of λ_i. We briefly indicate how to verify that the resulting defining equations are not isomorphic to each other.

6.1. $\mathscr{J}_1$.

$$y^2 = x\bigl(x^2 + (u^4+1)x + 4u^4\bigr), \tag{6.2}$$

$$J = \frac{1}{108}\frac{(u^8 - 10u^4 + 1)^3}{u^8(u^8 - 14u^4 + 1)}.$$

$$y^2 = x\bigl(x^2 + (u^4 + 6(2\omega+1)u^2 + 1)x - 32u^4\bigr), \tag{6.3}$$

$$J = \frac{1}{6912}\frac{\bigl(u^8 + 12(2\omega+1)u^6 - 10u^4 + 12(2\omega+1)u^2 + 1\bigr)^3}{u^8\bigl(u^8 + 12(2\omega+1)u^6 + 22u^4 + 12(2\omega+1)u^2 + 1\bigr)}.$$

Both the equations (6.2) and (6.3) have two I_8 fibers at $u = 0$ and ∞ and eight I_1 fibers. Suppose they define isomorphic elliptic curves over $k(u)$. Then there must be a linear transformation of u fixing 0 and ∞ which sends one J into the other, J denoting the classical absolute invariant of the generic fibre (normalized so that $J = 1$ for $y^2 = x^3 - x$). But this is impossible, as the positions of the eight I_1 fibers are determined by the simple poles of J and they cannot be transformed by such a linear transformation. This proves that the two elliptic fibrations are not isomorphic to each other.

6.2. $\mathscr{J}_2$**.**

$$y^2 = x\big(x^2 - (3u^4 + 6u^2 - 1)x + 32u^6\big), \tag{6.4}$$

$$J = \frac{1}{6912}\frac{(9u^8 - 60u^6 + 30u^4 - 12u^2 + 1)^3}{u^{12}(u^4 - 10u^2 + 1)(9u^4 - 2u^2 + 1)}.$$

6.3. $\mathscr{J}_3$**.**

$$y^2 = x^3 + u^4(u^4 + 1), \quad J = 0. \tag{6.5}$$

6.4. $\mathscr{J}_4$**.**

$$y^2 = x^3 - (u^3 - 1)^2 x, \ (u = x_1), \quad J = 1. \tag{6.6}$$

$$y^2 = x^3 - (v^3 - v)^3, \ (v = x_2), \quad J = 0. \tag{6.7}$$

These are the two obvious elliptic fibrations on S induced by the projections $C_1 \times C_2 \to C_1$ or C_2.

6.5. $\mathscr{J}_5$**.**

$$y^2 = x(x - 4)\big(x + 2u(u^2 + 3u + 3)\big). \tag{6.8}$$

$$J = \frac{1}{27}\frac{(u^6 + 6u^5 + 15u^4 + 20u^3 + 15u^2 + 6u + 4)^3}{u^2(u + 2)^2(u^2 + u + 1)^2(u^2 + 3u + 3)^2}.$$

6.6. $\mathscr{J}_6$**.**

$$y^2 = x(x + 2u^2)\big(x - u(u^2 - u + 1)\big), \tag{6.9}$$

$$J = \frac{1}{27}\frac{(u^4 + 5u^2 + 1)^3}{u^2(u^4 + u^2 + 1)^2}.$$

$$y^2 = x(x - \omega u^2)\big(x + u(2u - 1)(u + \omega^2)\big), \tag{6.10}$$

$$J = \frac{4}{27}\frac{\omega\big(2u^2 - (\omega + 2)u - \omega^2\big)^3\big(2u^2 - 2(\omega + 2)u - \omega^2\big)^3}{u^2(u - 1)^2(2u - 1)^2(u + \omega^2)^2(2u + \omega^2)^2}.$$

The Legendre parameters we employed for the first equation (6.9) are $\lambda_1 = -\omega, \lambda_2 = -1$, while those for the second one (6.10) are $\lambda_1 = -\omega, \lambda_2 = 2$. That the two equations define nonisomorphic elliptic fibrations can be checked in the same way as the case for $\mathscr{J}_1$ above.

6.7. $\mathscr{J}_7$.

$$y^2 = x\bigl(x^2 - u(u+1)(u+2)x + u^2(u+2)^2\bigr), \tag{6.11}$$

$$J = \frac{4}{27}\frac{(u^2+2u-2)^3}{(u-1)(u+3)}.$$

$$y^2 = x\bigl(x^2 + \omega u(u-1)(u-3\omega-2)x + 2\omega^2u^2(u-1)^2\bigr). \tag{6.12}$$

$$J = \frac{1}{27}\frac{(u^2-2(3\omega+2)u+3\omega-11)^3}{(u^2-2(3\omega+2)u+3\omega-13)}.$$

In this case, we can check that there is a linear transformation of u sending the first J into the second one. However it does not preserve the position of singular fibres which can be seen from the discriminants (but not from the absolute invariants). Hence (6.11) and (6.12) are not isomorphic.

6.8. $\mathscr{J}_8$.

$$y^2 = x\bigl(x^2 + u(3u+2)x + u^2(1+3u+3u^2+2u^3)\bigr), \tag{6.13}$$

$$J = \frac{4}{27}\frac{(6u^3-3u-1)^3}{u^2(2u+1)^2(u^2+u+1)^2(8u+3)}.$$

6.9. $\mathscr{J}_9$.

$$y^2 = x^3 + u(u^2-4)^3, \quad J = 0. \tag{6.14}$$

6.10. $\mathscr{J}_{10}$.

$$y^2 = x^3 + u^2(u+3)x^2 + u^2(-2u^2-2u+3)x + u^4(u-1). \tag{6.15}$$

We omit J.

6.11. $\mathscr{J}_{11}$.

$$\begin{aligned} y^2 = x^3 &- 27u^2(u^4+6u^3+5u^2-6u+1)x \\ &- 54u^3(u^2+1)(u^4+9u^3+20u^2-9u+1). \end{aligned} \tag{6.16}$$

$$\begin{aligned} y^2 = x^3 &- 27u^2(4u^4-12u^3+10(\omega+1)u^2-6\omega u+\omega)x \\ &+ 27u^3\bigl(16u^6-72u^5+6(19+10\omega)u^4 \\ &\quad - 63(1+2\omega)u^3+3(-9+10\omega)u^2+18u-2\bigr). \end{aligned} \tag{6.17}$$

We omit J, but it can be checked that the two elliptic fibrations are not isomorphic to each other by a similar argument as before.

Thus we have listed the defining equations of elliptic fibrations (with a section) on the Kummer surface $S = \mathrm{Km}(C_1 \times C_2)$ with C_i given by (6.1) over an algebraically closed field k of characteristic 0. Needless to say that the function field $k(x, y, u)$ defined by each of the equations (6.2) through

(6.17) is isomorphic to one and the same function field $k(S)$, which is the extension $k(x_1, x_2, t)$ with $t = y_1/y_2$ determined by (6.1).

7. Closing remark

In closing this paper, it should be remarked that the problems posed in the Introduction (§1.1) should be interesting and worth considering for more general $K3$ surfaces.

Even in the case of Kummer surfaces, we could ask such questions as follows:

PROBLEM 5. *Study Problems 1 and 2 for the Kummer surface* $X = \mathrm{Km}(A)$*, when* A *is the Jacobian variety of a genus two curve.*

For this, the so-called 16_6-configuration of thirty-two (-2)-curves on X should play an important role in place of the twenty-four basic curves used in this paper. A special case has been treated in Shioda [**13**].

According to Weil [**15**], a principally polarized abelian surface is either the Jacobian variety of a genus two curve or a product of two elliptic curves. Beyond the case of principally polarized abelian surfaces, we ask:

PROBLEM 6. *Find at least one elliptic parameter for the Kummer surface* $X = \mathrm{Km}(A)$ *when* A *is a generic member in a family of polarized abelian surfaces.*

The coefficients in the defining equation (especially the discriminant) for such should be related to some modular forms or theta-functions of interest.

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In answering these requests, (i) we have tried our best to minimize the possible errors (but still some errors might have crept in during correction). (ii) In char $k = 3$, it is possibe that some of the defining equations becomes a quasi-elliptic fibration. We should leave this question to the interested reader, but let us say this: if for example we let $\lambda_1 = \lambda_2 = -1$ in char $k = 3$, then $\mathscr{J}_3$-fibration gives the same equation as the $\mathscr{J}_3$-fibration in §6 (for

char $k \neq 3$), which is indeed quasi-elliptic in char $k = 3$. (iii) We have added a new section §6 in the revised version to respond to this request.

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